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PLANE HYDRODYNAMIC PROBLEM FOR VISCOELASTIC LUBRICATION

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The Reynolds theory of a one-dimensional hydrodynamic problem of lubrication, when the liquid is viscoelastic with an arbitrary relaxation core, is generalized. On the basis of a qualitative and numerical analysis, it is shown that the effect of relaxation leads to expansion of the pressure distribution and to a decrease in the carrying capacity of a lubricating layer.

In friction units with lubrication, situations can occur where a particle of a lubricant passes through the region of high gradients for a time comparable with a time of relaxation to local thermodynamic equilibrium. In this case, in calculations one must take into account the effects of time nonlocality (or, in other words, inheritance). The fact that viscoelastic properties of fluids have been studied well only under conditions of small departures from equilibrium hinders the solution of this problem. For friction units, when departures from equilibrium are large, it is not completely clear in which way retardation is to be taken into account in the rheological law. Generally speaking, the dependence of stresses on deformations in the form of a complex nonlinear functional is possible.

In the present work, a one-dimensional hydrodynamic problem for a viscoelastic incompressible lubricant in the absence of slip is considered. The generalization of the Reynolds theory for the simplest rheological law which allows for retardation and dependence of viscosity on pressure is constructed.

We assume that there is a steady-state isothermal flow in the region $h_1(x) \le y \le h_2(x)$, where $y = h_i(x)$ are the equations of the surfaces of rigid bodies which are in contact with the fluid. This problem was studied in detail for the Newtonian geometry [1–3]. Therefore, we take for granted a number of relations which were obtained previously for this problem [1–3] and are not related to the rheological law.

We take the continuity equation and the momentum equations in the form

$$\frac{d}{dx}\int_{h_1(x)}^{h_2(x)} u dy = 0 , \tag{1}$$

$$0 = -\partial_x p + \partial_x \tau_{xx} + \partial_y \tau_{xy} , \tag{2}$$

$$0 = -\partial_y p + \partial_x \tau_{yx} + \partial_y \tau_{yy} . \tag{3}$$

Here u is the x-component of the flow velocity. We adopt the following rheological law:

$$\tau_{xy}(t) = 2\mu(p(t)) \int K(t - t_0)\, e_{xy}(t_0)\, dt_0 , \tag{4}$$

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where μ is a positive function of pressure.

In (4), integration over time is performed in a particle of the medium (but not at the point of space!). The relaxation core is assumed to be independent of pressure. It is convenient to normalize the core by the condition

$$\int K(t)\,dt = 1\,. \tag{5}$$

Then the quantity μ can be interpreted as ordinary shear viscosity observed in slow flows of this fluid.

Retaining, in a standard manner [1–3], only the main terms in relations (2)–(4), we come to the equations

$$0 = -\partial_x p + \partial_y \tau_{xy}\,, \tag{6}$$

$$0 = -\partial_y p\,, \tag{7}$$

$$\tau_{xy}(t) = 2\mu\,(p(t)) \int K(t - t_0)\,\partial_y u\,(t_0)\,dt_0\,. \tag{8}$$

As in the classical approach [1–3], it follows from Eq. (7) that the pressure field depends only on the variable x.

Let us consider the functional form of the relaxation core. It is obvious that for the causality of the model to be provided, the condition $K(t) = 0$ for $t < 0$ must hold. Moreover, from general thermodynamic considerations [4, 5] there follows the condition on the Fourier transform of the core:

$$\operatorname{Re} K_{\mathrm{F}}(\omega) \geq 0\,. \tag{9}$$

Imposing an additional condition on the quantity $K(0)$, we can attain the finiteness of the velocity of propagation of shear disturbances [6, 7]:

$$0 < K(0) < +\infty\,. \tag{10}$$

In addition to conditions (5), (9), and (10), we adopt that $K(t)$ for $t \geq 0$ is a positive monotonically decreasing function (fading memory).

In practice, one makes wide use of cores of the form

$$K(t) = \sum_n A_n \tau_n^{-1} \exp(-t/\tau_n)\,, \quad t \geq 0\,, \tag{11}$$

where $\tau_n > 0$ is a discrete set of internal times of relaxation. The additional conditions

$$\sum_n A_n = 1\,, \quad A_n > 0\,, \quad \sum_n A_n \tau_n^{-1} < +\infty$$

are sufficient to satisfy (5), (9), and (10). The representation of (11) corresponds to the realization of the discrete set of dissipative relaxation processes in a particle of the fluid. A more general representation which covers the case of a continuous spectrum of dissipative internal relaxation processes has the form

$$K(t) = \int_{\tau>0} A(\tau)\,\tau^{-1} \exp(-t/\tau)\,d\tau\,, \quad t \geq 0\,. \tag{12}$$

In this case, to comply with (5), (9), and (10) the following conditions are sufficient:

$$\int\limits_{\tau>0} A(\tau)\, d\tau = 1\,, \quad A(\tau) > 0\,, \quad \int\limits_{\tau>0} A(\tau)\, \tau^{-1}\, d\tau < +\infty\,.$$

Core (11) is obtained from core (12) when the function $A(\tau)$ falls into the sum of δ-functions. From the physical studies [8] made for a number of one-component fluids it follows that the spectrum of relaxation times often contains two discrete points τ_1 and τ_2 and also a continuous section $0 < \tau < \tau_3$, with $\tau_1 > \tau_3$ and $\tau_2 > \tau_3$.

At the boundary of the flow region we adopt the ordinary no-slip conditions [1–3]:

$$u\big|_{y=h_1(x)} = U\,, \quad u\big|_{y=h_2(x)} = U\,, \tag{13}$$

where U is a positive quantity with the dimensions of velocity. Generally speaking, the boundary conditions for a hereditary fluid can have the form of relaxation equations with phenomenological parameters that characterize the interaction with a solid surface. However, in the present work we use only classical relations (13).

We assume that the velocity field is decomposed into

$$u = U + v\,, \quad |v| << U\,. \tag{14}$$

In the flow region we introduce the variable η, $0 \le \eta \le 1$, by the formula

$$y = h_1(x) + h(x)\,\eta\,, \quad h(x) = h_2(x) - h_1(x)\,.$$

Since the flow is steady, integration over time in relation (8) can be replaced by integration along the streamline. In the ordinary assumptions of lubrication theory, streamlines differ negligibly from the lines η = const. Thus, from (8), with account for (14), we obtain

$$\tau_{xy}(x, y) = \mu(p(x))\, U^{-1} \int\limits_{a}^{x_1} K((x - x_0)\, U^{-1})\, v_{,y}(x_0, y_*(x_0))\, dx_0 + g(x, y)\,. \tag{15}$$

Here $x = a$ is the left boundary of the flow region,

$$y_*(x_0) = h_1(x_0) + (y - h_1(x))\, h(x_0)\, h(x)^{-1}\,,$$

and the function $g(x, y)$ describes the effect of the state of the fluid prior to its entry into the lubrication region $x \ge a$:

$$g(x, y) = \mu(p(x))\, U^{-1} \int\limits_{x_0<a} K((x - x_0)\, U^{-1})\, v_{,y}(x_0, y(x_0))\, dx_0\,.$$

We assume that in the region $x < a$ the velocity field differs negligibly from the constant U, so that the function $g(x, y)$ on the right-hand side of expression (15) can be ignored. We seek the velocity profile in the following form:

$$v = f(x)\,(h_2(x) - y)\,(y - h_1(x))\,, \tag{16}$$

which corresponds to boundary conditions (13). Hence we calculate

$$v_{,y} = f(x)\,(h_2(x) - h_1(x))\,(1 - 2\eta)\,.$$

Substituting this expression into formula (15), we find the viscous stresses

$$\tau_{xy}(x, y) = \mu(p(x))\,(h_1(x) + h_2(x) - 2y)\,(Uh(x))^{-1} \int_a^x K((x - x_0)\,U^{-1})\,f(x_0)\,h(x_0)\,dx_0\,.$$

This equation, in combination with (6), allows one to obtain ordinary differential equations for pressure:

$$\mu(p)^{-1}\,p_{,x} = -\,2\,(Uh(x))^{-1} \int_a^x K((x - x_0)\,U^{-1})\,f(x_0)\,h(x_0)\,dx_0\,. \tag{17}$$

Then, substituting expressions (14) and (16) into the continuity equation (1), we derive the relation

$$\frac{d}{dx}\,(Uh + 6^{-1}\,fh^3) = 0\,,$$

which can be integrated:

$$Uh + 6^{-1}\,fh^3 = Q\,. \tag{18}$$

Here the integration constant Q has the meaning of ordinary flow rate. From (17) and (18) we find the final expression for pressure:

$$\mu(p)^{-1}\,p_{,x} = -\,12\,(Uh(x))^{-1} \int_a^x K((x - x_0)\,U^{-1})\,(Q - Uh(x_0))\,h(x_0)^{-2}\,dx_0\,. \tag{19}$$

As the boundary conditions for this equation we take the same relations as in the classical approach [1–3]:

$$p\,\big|_{x=a} = p_0\,,\quad p\,\big|_{x=b} = p_0\,,\quad p_{,x}\,\big|_{x=b} = 0\,, \tag{20}$$

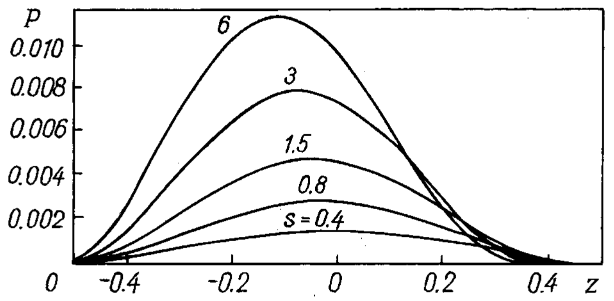
where $b > a$. At a given value of a, problem (19), (20) allows one to determine the distribution of pressure and the parameters Q and b.

We note that by virtue of Eq. (19) the equality

$$p_{,x}\,\big|_{x=a} = 0\,.$$

always holds.

The classical problem for a Newtonian lubricant [1–3] is obtained from (19) and (20) by formal substitution of the Dirac δ-function for the relaxation core. This note and an analysis of Eq. (19) make it possible to reveal qualitative differences from the case of the Newtonian lubricant. Thus, the extrema of the pressure field for the Newtonian lubricant coincide with zeros of the integrand in (19); for a viscoelastic lubricant they are shifted downstream relative to these zeros. If the function $h = h(x)$ is symmetric, i.e., $h(x) = h(-x)$, then the distribution of the pressure gradient $p_{,x}$ for the Newtonian lubricant is also symmetric; for a viscoelastic lubricant this property does not hold. An increase of the velocity U with a constant flow rate Q leads to enhancement of relaxation effects and an increase of the length $(b - a)$; in contrast, a decrease of the velocity U makes all the characteristics approach the values determined for the Newtonian lubricant.



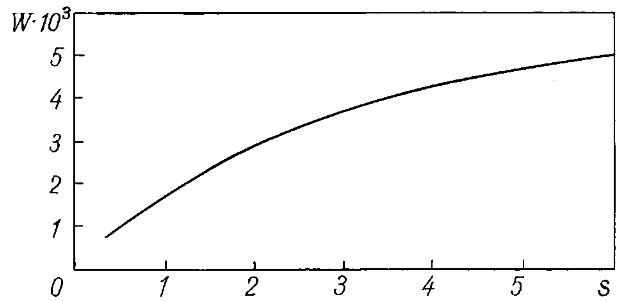


Fig. 1. Distribution of the dimensionless pressure for α = –0.5 at different values of the parameter *s*.

Fig. 2. Dimensionless carrying capacity of the lubricating layer as a function of the parameter *s*.

To obtain quantitative results, we consider a lubricant with constant viscosity for the parabolic dependence

$$h(x) = h_0 + \gamma x^2 .$$

We pass, in the ordinary way, to the dimensionless quantities [2]:

$$z = h_0^{-1/2}\gamma^{1/2} x , \quad \alpha = h_0^{-1/2}\gamma^{1/2} a , \quad \beta = h_0^{-1/2}\gamma^{1/2} b ,$$

$$q = (Uh_0)^{-1} Q , \quad P = h_0^{3/2}\gamma^{1/2}(12U\mu)^{-1} p .$$

Then Eq. (19) can be transformed to yield

$$P_{,z} = \tau_0 (1 + z^2)^{-1} \int_\alpha^z K((z - z_0)\tau_0)(z_0^2 - (q - 1))(1 + z_0^2)^{-2} dz_0 , \tag{21}$$

where $\tau_0 = h_0^{1/2}\gamma^{-1/2}U^{-1}$.

We recall that one of the conditions of applicability of the developed theory (14) requires the smallness of the dimensionless quantity $|q - 1|$.

We take the simplest expression for the core:

$$K(t) = \tau_1^{-1} \exp(-t/\tau_1) , \quad t \ge 0 ,$$

where $\tau_1 > 0$ is the relaxation time, and we denote $s = \tau_0/\tau_1$. We can obtain a numerical solution of problem (20), (21) at the given values of α and *s*. Figures 1 and 2 give the corresponding results for the distribution of the dimensionless pressure and for values of the quantity

$$W = \int_\alpha^\beta P dz ,$$

which characterizes the carrying capacity of the lubricating layer. The values α = –0.5 and $p_0 = 0$ were fixed, and the value of *s* changed from 0.4 to 6. For all variants, the calculated value of $|q - 1|$ did not exceed 10^{-1}. We note the decrease of the carrying capacity with decrease in the parameter *s*, i.e., with increase in the effect of relaxation.

Thus, under certain assumptions, the strongest of which is (14), we succeeded in obtaining a simple generalization of the Reynolds theory to the case of a viscoelastic lubricant. It is shown that relaxation effects lead, generally speaking, to "blurring" of the pressure profile and a decrease in the carrying capacity of the lubricating layer.

NOTATION

x and y, horizontal and vertical coordinates; t, time; $h_1(x)$ and $h_2(x)$, functions describing the shape of rigid bodies bounding the flow region; $h(x) = h_2(x) - h_1(x)$, value of the gap; u, horizontal component of the flow velocity; p, pressure; τ_{xx}, τ_{xy}, τ_{yx}, and τ_{yy}, components of the tensor of viscous stresses; e_{xy}, component of the tensor of deformation rates; $K = K(t)$, relaxation core; μ, shear viscosity; $K_F(\omega)$, Fourier transform of the core; τ_n, $n = 1, 2, ...$, internal times of relaxation; A_n and $A(\tau)$, weighting factors characterizing the contribution of internal relaxation processes; U, velocity at the boundary of the flow; $v = u - U$, function with the dimensions of velocity; η, auxiliary dimensionless parameter; a and b, left and right boundaries of the flow region; $g(x, y)$, function characterizing the effect of the region $x < a$ on viscous stresses; t_0, auxiliary variable of integration over time; x_0 and z_0, auxiliary variables of integration over the abscissa; $f(x)$, auxiliary function characterizing the value of velocity disturbances; Q, volumetric flow rate; p_0, pressure in the environment (e.g., atmospheric pressure); h_0 and γ, parameters in the parabolic dependence $h(x)$; z, α, and β, dimensionless values of the coordinate; q, dimensionless flow rate; P, dimensionless pressure; τ_0, auxiliary constant with the dimensions of time; s, auxiliary dimensionless quantity; W, dimensionless carrying capacity of the lubricating layer.

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